

Optimizing over iid distributions and the Beat the Average game

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Based on joint work with Pierre C Bellec

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Musings on large numbers

Theorem (Law of large numbers)

Let $X_1, X_2, \dots$ be iid nonnegative random variables with finite first moment,

$$\mathbb{E}[X_1] < \infty.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = \mathbb{E}[X_1]$$

with probability 1.

Intuition: the sample average

$$\frac{1}{n} \sum_{i=1}^n X_i$$

does not fluctuate much with n .

Musings on large numbers

- ▶ What can we say **without** assuming finite first moment?
- ▶ For example, can we bound

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^n X_i > \frac{\alpha}{m}\sum_{i=1}^m X_i\right]$$

for $n > m$ and $\alpha > 0$?

- ▶ Let's look at two special cases:

$$\mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right], \quad \mathbb{P}[X_1 + X_2 + X_3 < 2X_4].$$

Beat the Average

- ▷ In the casino, you can play a new game: you roll a die and the game master rolls two. You win if your number is *at least* the average of the game master's numbers.
- ▷ One round costs 1 €, and you get 2 € back if you win.
- ▷ Tricky point: the three dice are identical, but exotic!



Beat the Average

- ▷ With standard dice, you should play: by symmetry,

$$\mathbb{P}\left[X_1 > \frac{X_2 + X_3}{2}\right] = \mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right],$$

and there's a nonzero chance of $X_1 = \frac{X_2 + X_3}{2}$. Therefore

$$\mathbb{P}[\text{win}] > \frac{1}{2}.$$

- ▷ But the dice do not look standard!
- ▷ To assume the worst case, you need to determine how large your losing probability can get,

$$\sup_{\mu} \mathbb{P}_{\mu^{\otimes 3}}\left[X_1 < \frac{X_2 + X_3}{2}\right] = ?$$

Beat the Average, card version

- ▶ The casino also offers a version with cards: you draw one card from a shuffled deck and the game master draws two. You win if your card is at least the average of the game master's.
- ▶ Suppose that the deck has n cards. Then:

Theorem

- (a) For any n , the casino can trick you at least as much in the card version as in the dice version.
- (b) For $n \rightarrow \infty$, the maximal losing probabilities coincide.

For the proof of (a), use a *random* deck!

Beat the Average, card version

Taking $n = 3$ shows that

$$\mathbb{P}\left[X_1 < \frac{X_2 + X_3}{2}\right] \leq \frac{2}{3}.$$

Moreover, $\frac{2}{3}$ is achieved for any n by the deck with values

$$1, 2, 4, \dots, 2^{n-1}.$$

Corollary

$$\sup_{\mu} \mathbb{P}_{\mu^{\otimes 3}}\left[X_1 < \frac{X_2 + X_3}{2}\right] = \frac{2}{3}$$

$\Rightarrow$ You should play neither the card nor the dice game!

Optimizing over iid distributions, harder case

Theorem

$$0.400695 \leq \sup_{\mu} \mathbb{P}_{\mu^{\otimes 4}}[X_1 + X_2 + X_3 < 2X_4] \leq 0.417$$

- ▷ We conjecture that the explicit construction behind our lower bound is optimal.
- ▷ Let's focus on the upper bound.

Harder case

- ▶ The “card version” of this problem still gives a convergent hierarchy of upper bounds.

- ▶ A “deck of cards” is determined by values

$$X_1, \dots, X_n \in \mathbb{R}_+.$$

- ▶ An optimal deck is such that the inequality

$$X_i + X_j + X_k < 2X_\ell$$

is satisfied for as many tuples $ijkl \in \{1, \dots, n\}$ as possible.

- ▶ This is a problem of finding a **maximal feasible subsystem** of inequalities.

Determining a maximal feasible subsystem

To approach it, solve the following mixed integer linear program:

$$\begin{array}{ll}\text{maximize} & \sum_{ijkl} y_{ijkl} \\ \text{with respect to} & y_{ijkl} \in \{0, 1\} \\ & x_i \in [0, 1] \\ \text{subject to} & x_i + x_j + x_k + M(y_{ijkl} - 1) \leq 2x_\ell - \varepsilon \\ & 0 \leq x_1 \leq \dots \leq x_n\end{array}$$

Here, $\varepsilon \ll 1$ and $M := 3 + \varepsilon$ are constants.

Idea: $y_{ijkl} = 1$ indicates that the inequality $x_i + x_j + x_k < 2x_\ell$ is satisfied.

Determining a maximal feasible subsystem

- ▷ Problem: a numerical solution does not give a rigorous certificate of optimality.
- ▷ Alternative approach: note that

$$\begin{aligned} X_1 + X_2 + X_5 < 2X_3 \text{ is incompatible with} \\ X_3 + X_4 + X_7 < 2X_5 \end{aligned}$$

and

$$\begin{aligned} X_1 + X_2 + X_5 < 2X_4 \text{ is incompatible with} \\ X_1 + X_4 + X_6 < 2X_5 \\ X_3 + X_6 + X_8 < 2X_7 \\ X_4 + X_7 + X_9 < 2X_8. \end{aligned}$$

etc.

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etc.

- ▷ Principle: If we have a collection of N *disjoint* incompatible systems, then at least N inequalities must be violated overall.

A second mixed integer linear program

- ▶ We can search for such disjoint collections of systems using a *second* mixed integer linear program!
- ▶ To formulate it, let

$$\mathcal{T}^s := \{ijkl \mid y_{ijkl} = 1\},$$

$$\mathcal{T}^v := \{ijkl \mid y_{ijkl} = 0\},$$

where the y_{ijkl} are from the first program.

- ▶ This will guide us in finding such disjoint collections of systems.

A second mixed integer linear program

This idea can be formalized as follows:

find any feasible $y_{tu} \in \{0, 1\}$ with $(t, u) \in \mathcal{T}^s \times \mathcal{T}^v$,

subject to $\sum_{u \in \mathcal{T}^v} y_{tu} \leq 1$ with $t \in \mathcal{T}^s$,

$$F(q, u) + \sum_{t \in \mathcal{T}^s} F(q, t) y_{tu} \geq 0 \quad \text{with } q \in [m], u \in \mathcal{T}^v.$$

Here, we write

$$F(q, t) := \delta_{q \leq i} + \delta_{q \leq j} + \delta_{q \leq k} - 2\delta_{q \leq \ell}.$$

A second mixed integer linear program

For $n = 20$, this terminates and we get:

- ▷ There are 19380 candidate inequalities.
- ▷ Half of them already contradict $0 \leq x_1 \leq \dots \leq x_{20}$.
- ▷ Among the remaining 9690, we find 1614 disjoint incompatible subsystems.
- ▷ For each, incompatibility follows by simply adding up the inequalities.
- ▷ This gives the rigorous upper bound of

$$\frac{9690 - 1614}{19380} = 0.4167 \dots$$

Lower bound

Consider more generally the maximal probability of an inequality $\sum_i c_i X_i > 0$ under iid distributions.

Proposition

For every ν and $\varepsilon > 0$, there is μ such that

$$\mathbb{P}_{\mu^{\otimes n}} \left[\sum_i c_i X_i > 0 \right] \geq \frac{\mathbb{P}_{\nu^{\otimes n}} [\sum_i c_i X_i > 0]}{1 - \mathbb{P}_{\nu^{\otimes n}} [\sum_i c_i X_i = 0]} - \varepsilon.$$

For the proof, take

$$Y_i := X_{i,1} + \eta X_{i,2} + \eta^2 X_{i,3} + \dots$$

for $\eta \ll 1$ and the $X_{i,j}$ iid with law ν .

Lower bound

Applying the proposition with

$$\nu := p\delta_0 + \frac{1-p}{N} \sum_{i=1}^N \delta_{1-2^{-i}}$$

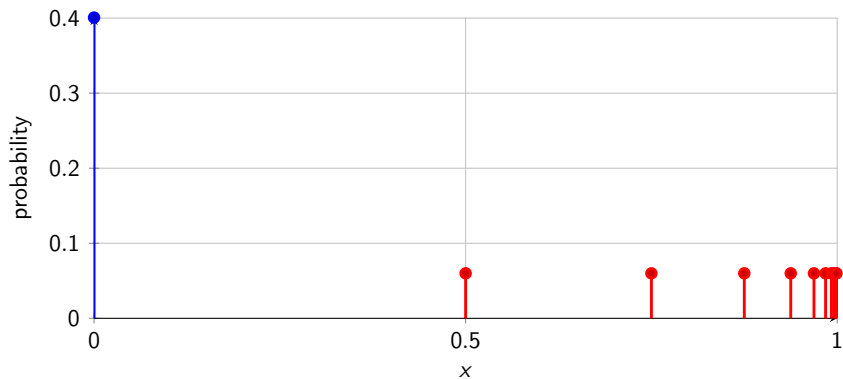
for $N \gg 1$ results in

$$\sup_{\mu} \mathbb{P}_{\mu^{\otimes n}}[X_1 + X_2 + X_3 < 2X_4] \geq \sup_{p \in (0,1)} \frac{p(2-p)}{1+p+p^2+p^3}.$$

The maximum ≈ 0.400695 is achieved at $p \approx 0.474346$.

Lower bound

Discrete measure ν on $[0, 1]$ for $N = 10$



AlphaEvolve

MATHEMATICAL EXPLORATION AND DISCOVERY AT SCALE

BOGDAN GEORGIEV, JAVIER GÓMEZ-SERRANO, TERENCE TAO, AND ADAM ZSOLT WAGNER

- ▷ AlphaEvolve is a coding agent that automatically evolves code for optimization problems.
- ▷ Tested on 67 representative mathematical optimization problems.
- ▷ In many cases (e.g. kissing numbers), existing bounds were improved.
- ▷ For our $\mathbb{P}[X_1 + X_2 + X_3 < 2X_4]$, it found a lower bound of 0.389.

New result distribution

Visualization of results across 67 problems.

■ New result ■ Former new result, got improved upon ■ Worse than literature bound
■ Matched known optimal bound ■ Matched literature bound / N/A



Summary

- ▶ Motivated by the law of large numbers, we have studied how large probabilities like

$$\mathbb{P}\left[\sum_i c_i X_i > 0\right]$$

can get under an iid distribution.

- ▶ We have methods to compute upper and lower bounds, but computing exact values is still open.
- ▶ Conjecture: an exact computation method based on the fractal nature of near-optimal distributions exists.
- ▶ Our $\mathbb{P}[X_1 + X_2 + X_3 < 2X_4]$ has become a benchmark problem for AI mathematics.